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SOLUTION FOR FUZZY DIFFERENTIAL EQUATIONS USING FOURTH ORDER RUNGE-KUTTA METHOD WITH EMBEDDED HARMONIC MEAN

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ABSTRACT

In this paper, an attempt has been made to determine a numerical solution for the first order fuzzy differential equations by using fourth order Runge-kutta embedded harmonic mean. The accuracy and applicability of the proposed method is illustrated by solving a fuzzy initial value problem with triangular fuzzy number.

KEYWORDS: Fuzzy Differential Equations, Runge-kutta fourth order method, Embedded Harmonic Mean, Triangular Fuzzy Number.

INTRODUCTION

The theory of fuzzy differential equation plays an important role in modelling of science and engineering problems because this theory represents a natural way to model dynamical systems under uncertainty.The applicability of the fuzzy differential equation leads to a several number of research works in the open literature. First order linear fuzzy differential equation is one of the simplest fuzzy differential equation, which appear in many applications. Some of the reviewed research papers are cited below for better understanding of the present paper.The concept of fuzzy derivative was first introduced by S.L.Chang and L.A.Zadeh in [6]. D.Dubois and Prade [7] discussed differentiation with fuzzy features. M.L.puri, D.A.Ralescu [24] and R.Goetschel , W.Voxman [10] contributed towards the differential of fuzzy functions.The fuzzy differential equation and initial value problems were extensively studied by O.Kaleva [15,16] and by S.Seikkala [25].Recently many research papers are focused on numerical solution of fuzzy initial value problems(FIVPS) . Numerical Solution of fuzzy differential equation has been introduced by M.Ma, M.Friedman, A.Kandel [18] through Euler method and by S.Abbasbandy and T.Allahviranloo [1] by taylor method.Runge–Kutta methods have also been studied by authors [2,22]. V.Nirmala,N.Saveetha,S.Chenthurpandiyan discussed on numerical solution of fuzzy differential equation by runge-kutta method with higher order derivative approximations [21]. R.Gethsi Sharmila and E.C.Henry Amirtharaj discussed on numerical Solutions of first order fuzzy initial value problems by non-linear trapezoidal formulae based on variety of Means[13].A new Fourth order runge-kutta method with embedded harmonic mean for initial value problems was proposed by Nazeerudin Yaacob and Bahrom Sanugi[19], also it was studied by R.Ponalagusamy and S.Senthilkumar[23].Followed by the introduction this paper is organized as follows:In section 2, some basic results of fuzzy numbers and definitions of fuzzy derivative are given. In section 3, the fuzzy initial value problem has been discussed. Section 4 describes the general structure of the fourth order runge-kutta embedded harmonic mean method,In section 5 the runge kutta fourth order embedded harmonic mean in particular, for solving fuzzy initial value problem has been discussed. Finally the applicability of the method is demonstrated by determining the numerical solution of the problem by applying the proposed method.

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PRELIMINARIES

Definition:(FUZZY NUMBER)

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r),\overline{u}(r))$ for all $r\in[0,1]$ which satisfy the following conditions.

i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$ with respect to any r.

ii) $\overline{u}(r)$ is a bounded right continuous non-decreasing function over $[0,1]$ with respect to any r.

iii) $(\underline{u}(r)\le\overline{u}(r))$ for all $r\in[0,1]$ then the r-level set is $[u]_r=\{x\backslash u(x)\ge r\};0\le r\le 1$

Clearly, $[u]_0=\{x\backslash u(x)\ge 0\}$ is compact, which is a closed bounded interval and we denote by $[u]_r=(\underline{u}(r),\overline{u}(r))$

Definition: (TRIANGULAR FUZZY NUMBER)

A triangular fuzzy number 'u' is a fuzzy set in E that is characterized by an ordered triple $(u_l,u_c,u_r)\in R^3$ with $u_l\le u_c\le u_r$ such that $[u]_0=[u_l;u_r]$ and $[u]_l=\{u_c\}$.

The membership function of the triangular fuzzy number 'u' is given by

$$u(x)=\begin{cases}\dfrac{x-u_l}{u_c-u_l} & ; \quad u_l\le x\le u_c\\ 1 & ; \quad x=u_c\\ \dfrac{u_r-x}{u_r-u_c} & ; \quad u_c\le x\le u_r\end{cases}$$

. we have :

(1) $u>0 \ if \ u_l > 0$

(2) $u\ge 0 \ if \ u_l\ge 0$

(3) $u<0 \ if \ u_c<0$ and

(4) $u\le 0 \ if \ u_c\le 0$.

Definition: (α - Level Set)

Let I be the real interval. A mapping $y:I\to E$ is called a fuzzy process and its α - *level Set* is denoted by $[y(t)]_\propto=[\underline{y}(t;\alpha),\overline{y}(t;\alpha)]$, $t\in I, 0<\propto<1$

Definition: (Seikkala Derivative)

The Seikkala derivative $y'(t)$ of a fuzzy process is defined by $[y'(t)]_\propto=[\underline{y}'(t;\alpha),\overline{y}'(t;\alpha)]$ $t\in I$, $0<\propto\le 1$ provided that this equation defines a fuzzy number, as in [25]

Lemma:

If the sequence of non-negative number $\{W_n\}_{n=0}^{m}$ satisfy $|Wn+1|\le A|Wn|+B$, $0\le n\le N-1$ for the given positive constants *A* and *B*, then $|Wn|\le A^n|W_0|+B\dfrac{A^n-1}{A-1}$, $0\le n\le N$

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Lemma

If the sequence of non-negative numbers $\{W_n\}_{n=0}^{m}$, $\{V_n\}_{n=0}^{N}$ satisfy $|W_{n+1}|\le|W_n|+A\,max\{|W_n|,|V_n|\}+B$, $|V_{n+1}|\le|V_n|+A\,max\{|W_n|,|V_n|\}+B$ for the given positive constants A and B, then $U_n=|W_n|+|V_n|$, $0\le n\le N$ we have, $U_n\le\overline{A}^nU_0+\overline{B}\frac{\overline{A}^n-1}{\overline{A}-1}$ $0\le n\le N$ where $\overline{A}=1+2A$ and $\overline{B}=2B$.

Lemma

Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^{'}(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed r, $0\le r\le1$, $D(y(t_{n+1}),y^0(t_{n+1}))\le h^2L(1+2C)$ where L is a bound of partial derivatives of F and G, and $C=Max\left\{\left|G\left[t_N,\underline{y}(t_N;r),\overline{y}(t_{N-1};r)\right]\right|,r\in[0,1]\right\}<\infty$.

Theorem

Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^{'}(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed $r, 0\le r\le1$, the numerical solutions of $\underline{y}(t_{n+1};r)$ and $\overline{y}(t_{n+1};r)$ converge to the exact solutions $\underline{Y}(t_{n+1};r)$ and $\overline{Y}(t_{n+1};r)$ uniformly in t.

Theorem

Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^{'}(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh<1$.Then for arbitrarily fixed $0\le r\le1$,the iterative numerical solutions of $\underline{y}^{(j)}(t_n;r)$ and $\overline{y}^{(j)}(t_n;r)$ converge to the numerical solutions $\underline{y}(t_n;r)$ and $\overline{y}(t_n;r)$ in $t_0\le t_n\le t_N$, when $j\to\infty$.

FUZZY INITIAL VALUE PROBLEM

Consider a first-order fuzzy initial value problem

$$\begin{cases} y'(t)=f(t,y(t)),t\in[t_0,T] \\ y(t_0)=y_0 \end{cases} \tag{3.1}$$

where y is a fuzzy function of t, $f(t,y)$ is a fuzzy function of the crisp variable $'t'$ and the fuzzy variable y. y' is the fuzzy derivative of 'y' and $y(t_0)=y_0$ is a triangular or a triangular shaped fuzzy number. We denote the fuzzy function 'y' by $y=[\underline{y},\overline{y}]$.It means that the r-level set of $y(t)$ for $t\in[t_0,T]$ is

$[y(t)]_r=[\underline{y}(t;r),\overline{y}(t;r)]$,

$[y(t_0)]_r=[\underline{y}(t_0;r),\overline{y}(t_0;r)],\quad r\in(0,1]$,

we write $f(t,y)=[\underline{f}(t,y),\overline{f}(t,y)]$ and

$\underline{f}(t,y)=F[t,\underline{y},\overline{y}],\quad \overline{f}(t,y)=G[t,\underline{y},\overline{y}]$,

because of $y'=f(t,y)$ we have

$$\underline{f}(t,y(t);r)=F[t,\underline{y}(t;r),\overline{y}(t;r)], \tag{3.2}$$

$$\overline{f}(t,y(t);r)=G[t,\underline{y}(t;r),\overline{y}(t;r)] \tag{3.3}$$

by using the extension principle, we have the membership function

$$f(t, y(t))(s) = sup\{y(t)(\tau) \setminus s = f(t,\tau)\},\ s \in R \tag{3.4}$$

so the fuzzy number $f(t, y(t))$ follows that $\left[f(t, y(t))\right]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)],\ r \in (0,1]$ (3.5)

where $\underline{f}\ (t, y(t); r) = min\{f(t,u) \,|\, u\left[y(t)\right]_r\}$ (3.6)

$$\overline{f}\left(t, y(t); r\right) = max\{f(t,u) \,|\, u\left[y(t)\right]_r\} \tag{3.7}$$

Definition 3.1 A function $f : R \to R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\varepsilon > 0$, $\delta > 0$ such that $\left|t - to\right| < \delta \Rightarrow D\left[f(t), f(t_0)\right] < \varepsilon$ exists. The fuzzy function considered are continuous in metric D and the continuity of $f(t, y(t); r)$ guarantees the existence of the definition of $f(t, y(t); r)$ for $t \in \left[t_0, T\right]$ and $r \in \left[0,1\right]$ [10]. Therefore, the functions *G* and *F* can be definite too.

FOURTH ORDER RUNGE KUTTA METHOD WITH EMBEDDED HARMONIC MEAN

The Fourth order Runge-kutta method with Embedded Harmonic Mean is a Runge-kutta method for approximating the solution of the initial value problem, $y'(t) = f(t, y(t))\ \ y(t_0) = y_0$.

The basis of all Runge-Kutta methods is to express the difference between the value of 'y' at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=0}^{m} w_i k_i \tag{4.1}$$

where w_i's are constant for all i and $k_i = hf(t_n + a_i h, y_n + \sum_{j=1}^{i-1} c_{ij} k_j)$ (4.2)

Increasing the order of accuracy of the Runge-Kutta methods,it has been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required[5].The method proposed by Goeken.D and Johnson.O[9] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta k_i terms (i>0) to obtain a higher order of accuracy without a corresponding increase in evaluations of 'f ', but with the addition of evaluations of f'.

The fourth order Runge-kutta method with embedded harmonic mean for step n+1 which was proposed by Nazeerudin Yaacob and Bahrom Sanugi[19].

$$y(t_{n+1}) = y(t_n) + h\left[\frac{k_2}{6} + \frac{k_3}{6} + \frac{2}{3}\left(\frac{k_1 k_2}{k_1 + k_2}\right) + \frac{2}{3}\left(\frac{k_3 k_4}{k_3 + k_4}\right)\right] \tag{4.3}$$

Where $k_1 = hf\left(t_n, y(t_n)\right)$ (4.4)

$$k_2 = hf\left(t_n + a_1 h, y(t_n) + a_1 h k_1\right) \tag{4.5}$$

$$k_3 = hf\left(t_n + (a_2 + a_3)h, y(t_n) + a_2 h k_1 + a_3 h k_2\right) \tag{4.6}$$

$$k_4 = hf\left(t_n + (a_4 + a_5 + a_6)h, y(t_n) + a_4 h k_1 + a_5 h k_2 + a_6 h k_3\right) \tag{4.7}$$

and the parameters $a_1, a_2, a_3, a_4, a_5, a_6$ are chosen to make y_{n+1} closer to y(t_{n+1}) .The value of parameters are

$$a_1 = \frac{1}{2}, a_2 = \frac{-1}{8}, a_3 = \frac{5}{8}, a_4 = \frac{-1}{4}, a_5 = \frac{7}{20}, a_6 = \frac{9}{10}$$

FOURTH ORDER RUNGE-KUTTA METHOD WITH EMBEDDED HARMONIC MEAN FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

Let the exact solution $[Y(t)]_r = [\underline{Y}(t;\ r), \overline{Y}(t;\ r)]$, is approximated by some $[y(t)]_r = [\underline{y}(t;\ r), \overline{y}(t;\ r)]$.

The grid points at which the solutions is calculated are $h = \dfrac{T - t_0}{N}$, $t_i = t_0 + ih; 0 \le i \le N$

From 4.3 to 4.7 we define

$$\underline{y}(t_{n+1}, r) - \underline{y}(t_n, r) = h\left[\frac{\underline{k_2}(t_n, y(t_n, r))}{6} + \frac{\underline{k_3}(t_n, y(t_n, r))}{6} + \frac{2}{3}\left[\frac{\underline{k_1}(t_n, y(t_n, r))\underline{k_2}(t_n, y(t_n, r))}{\underline{k_1}(t_n, y(t_n, r)) + \underline{k_2}(t_n, y(t_n, r))}\right] + \frac{2}{3}\left[\frac{\underline{k_3}(t_n, y(t_n, r))\underline{k_4}(t_n, y(t_n, r))}{\underline{k_3}(t_n, y(t_n, r)) + \underline{k_4}(t_n, y(t_n, r))}\right]\right] \quad (5.1)$$

where $k_1 = hF[t_n, \underline{y}(t_n, r), \overline{y}(t_n, r)]$ (5.2)

$$k_2 = hF[t_n + \frac{h}{2}, \underline{y}(t_n, r) + \frac{h}{2}\underline{k_1}(t_n, y(t_n, r)),\ \overline{y}(t_n, r) + \frac{h}{2}\overline{k_1}(t_n, y(t_n, r))] \quad (5.3)$$

$$\begin{aligned} k_3 = hF[t_n + \frac{h}{2},\ & \underline{y}(t_n, r) - \frac{h}{8}\underline{k_1}(t_n, y(t_n, r)) + \frac{5h}{8}\underline{k_2}(t_n, y(t_n, r)), \\ & \overline{y}(t_n, r) - \frac{h}{8}\overline{k_1}(t_n, y(t_n, r)) + \frac{5h}{8}\overline{k_2}(t_n, y(t_n, r))] \end{aligned} \quad (5.4)$$

$$\begin{aligned} k_4 = hF[t_n + h,\ & \underline{y}(t_n, r) - \frac{h}{4}\underline{k_1}(t_n, y(t_n, r)) + \frac{7h}{20}\underline{k_2}(t_n, y(t_n, r)) + \frac{9h}{10}\underline{k_3}(t_n, y(t_n, r)), \\ & \overline{y}(t_n, r) - \frac{h}{4}\overline{k_1}(t_n, y(t_n, r)) + \frac{7h}{20}\overline{k_2}(t_n, y(t_n, r)) + \frac{9h}{10}\overline{k_3}(t_n, y(t_n, r))] \end{aligned} \quad (5.5)$$

and

$$\overline{y}(t_{n+1}, r) - \overline{y}(t_n, r) = h\left[\frac{\overline{k_2}(t_n, y(t_n, r))}{6} + \frac{\overline{k_3}(t_n, y(t_n, r))}{6} + \frac{2}{3}\left[\frac{\overline{k_1}(t_n, y(t_n, r))\overline{k_2}(t_n, y(t_n, r))}{\overline{k_1}(t_n, y(t_n, r)) + \overline{k_2}(t_n, y(t_n, r))}\right] + \frac{2}{3}\left[\frac{\overline{k_3}(t_n, y(t_n, r))\overline{k_4}(t_n, y(t_n, r))}{\overline{k_3}(t_n, y(t_n, r)) + \overline{k_4}(t_n, y(t_n, r))}\right]\right] \quad (5.6)$$

Where

$$k_1 = hG[t_n, \underline{y}(t_n, r), \overline{y}(t_n, r)] \quad (5.7)$$

$$k_2 = hG[t_n + \frac{h}{2},\ \underline{y}(t_n, r) + \frac{h}{2}\underline{k_1}(t_n, y(t_n, r)), \overline{y}(t_n, r) + \frac{h}{2}\underline{k_1}(t_n, y(t_n, r))] \quad (5.8)$$

$$\begin{aligned} k_3 = hG[t_n + \frac{h}{2},\ & \underline{y}(t_n, r) - \frac{h}{8}\underline{k_1}(t_n, y(t_n, r)) + \frac{5h}{8}\underline{k_2}(t_n, y(t_n, r)), \\ & \overline{y}(t_n, r) - \frac{h}{8}\overline{k_1}(t_n, y(t_n, r)) + \frac{5h}{8}\overline{k_2}(t_n, y(t_n, r))] \end{aligned} \quad (5.9)$$

$$k_4 = hG[t_n + h,\ \underline{y}(t_n,r) - \frac{h}{4}\underline{k}_1(t_n, y(t_n,r)) + \frac{7h}{20}\underline{k}_2(t_n, y(t_n,r)) + \frac{9h}{10}\underline{k}_3(t_n, y(t_n,r)),$$
$$\overline{y}(t_n,r) - \frac{h}{4}\overline{k}_1(t_n, y(t_n,r)) + \frac{7h}{20}\overline{k}_2(t_n, y(t_n,r)) + \frac{9h}{10}\overline{k}_3(t_n, y(t_n,r))] \tag{5.10}$$

we define $$F\left[t_n, y(t_n,r)\right] = h\left[\frac{\underline{k}_2(t_n, y(t_n,r))}{6} + \frac{\underline{k}_3(t_n, y(t_n,r))}{6} + \frac{2}{3}\left[\frac{\underline{k}_1(t_n, y(t_n,r))\underline{k}_2(t_n, y(t_n,r))}{\underline{k}_1(t_n, y(t_n,r)) + \underline{k}_2(t_n, y(t_n,r))}\right] + \frac{2}{3}\left[\frac{\underline{k}_3(t_n, y(t_n,r))\underline{k}_4(t_n, y(t_n,r))}{\underline{k}_3(t_n, y(t_n,r)) + \underline{k}_4(t_n, y(t_n,r))}\right]\right] \tag{5.11}$$

$$G\left[t_n, y(t_n,r)\right] = h\left[\frac{\overline{k}_2(t_n, y(t_n,r))}{6} + \frac{\overline{k}_3(t_n, y(t_n,r))}{6} + \frac{2}{3}\left[\frac{\overline{k}_1(t_n, y(t_n,r))\overline{k}_2(t_n, y(t_n,r))}{\overline{k}_1(t_n, y(t_n,r)) + \overline{k}_2(t_n, y(t_n,r))}\right] + \frac{2}{3}\left[\frac{\overline{k}_3(t_n, y(t_n,r))\overline{k}_4(t_n, y(t_n,r))}{\overline{k}_3(t_n, y(t_n,r)) + \overline{k}_4(t_n, y(t_n,r))}\right]\right] \tag{5.12}$$

Therefore we have

$$\underline{Y}(t_{n+1},r) = \underline{Y}(t_n, r) + F[t_n, Y(t_n, r)]]$$
$$\overline{Y}(t_{n+1},r) = \overline{Y}(t_n, r) + G[t_n, Y(t_n, r)] \tag{5.13}$$

And

$$\underline{y}(t_{n+1},r) = \underline{y}(t_n, r) + F[t_n, y(t_n, r)]] \tag{5.14}$$
$$\overline{y}(t_{n+1},r) = \overline{y}(t_n, r) + G[t_n, y(t_n, r)]$$

Clearly $\underline{y}(t;r)$ and $\overline{y}(t;r)$ converge to $\underline{Y}(t;r)$ *and* $\overline{Y}(t;r)$ whenever $h \to 0$

NUMERICAL EXAMPLE

Consider fuzzy initial value problem

$$\begin{cases} y'(t) = y(t), & t \geq 0 \\ y(0) = (0.96 + 0.04r, 1.01 - 0.01r) \end{cases} \tag{6.1}$$

The exact solution is given by

$$Y(t,r) = [(0.96 + 0.04r)e^t, (1.01 - 0.01r)e^t]$$

At $t = 1$ we get $Y(1,r) = [(0.96 + 0.04r)e, (1.01 - 0.01r)e]$, $0 \leq r \leq 1$

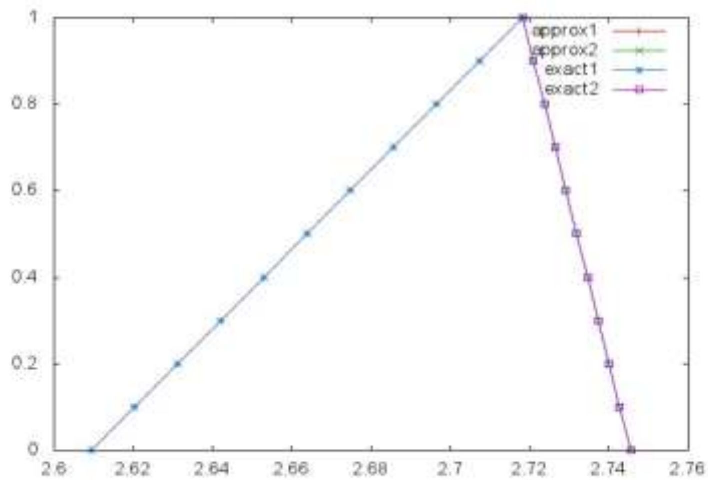
The values of exact and approximate solution with h= 0.1 is given in Table : I. The exact and approximate solutions obtained by the proposed method is plotted in Fig:1and the error1 and error2 is plotted in Fig:2.

Table:1 Exact and Approximate Solution

R	Exact Solution t=1	Approximate Solution (h=0.1)	Error 1	Error 2
	$\underline{Y}(t,r)$ $\overline{Y}(t,r)$	$\underline{y}(t,r)$ $\overline{y}(t,r)$		
0	2.609545 , 2.745459	2.609551 , 2.745465	5.522345e-006	5.809967e-006

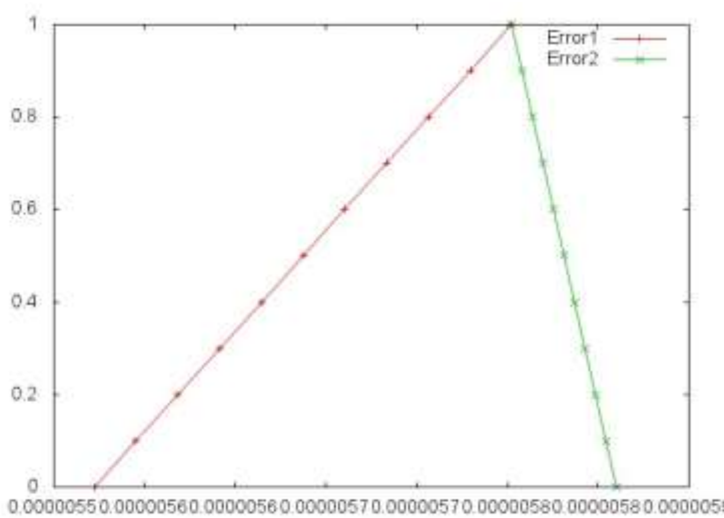
0.1	2.620418 , 2.742741	2.620424 , 2.742746	5.545355e-006	5.804215e-006
0.2	2.631291 , 2.740022	2.631297 , 2.740028	5.568364e-006	5.798462e-006
0.3	2.642164 , 2.737304	2.642170 , 2.737310	5.591374e-006	5.792710e-006
0.4	2.653037 , 2.734586	2.653043 , 2.734592	5.614384e-006	5.786957e-006
0.5	2.663911 , 2.731867	2.663916 , 2.731873	5.637394e-006	5.781205e-006
0.6	2.674784 , 2.729149	2.674789 , 2.729155	5.660404e-006	5.775452e-006
0.7	2.685657 , 2.726431	2.685662 , 2.726437	5.683413e-006	5.769700e-006
0.8	2.696530 , 2.723713	2.696536 , 2.723718	5.706423e-006	5.763947e-006
0.9	2.707403 , 2.720994	2.707409 , 2.721000	5.729433e-006	5.758195e-006
1.0	2.718276 , 2.718276	2.718282 , 2.718282	5.752443e-006	5.752443e-006

Figure:1



Exact & Approximate Solutions

Figure:2



Error 1 and Error 2

CONCLUSION

In this paper the fourth order Runge-Kutta method with embedded harmonic mean has being applied for finding the numerical solution of first order fuzzy differential equations using triangular fuzzy number.The efficiency and the accuracy of the proposed method have been illustrated by a suitable example. From the numerical example it has been observed that the discreet solutions by the proposed method almost coincide with the exact solutions.

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